CANADIAN THESES ON MICROFICHE

THÈSES CANADIENNES SUR MICROFICHE

National Library of Canada
Collections Development Branch

Canadian Theses on
Microfiche Service

Ottawa, Canada
K1A 0N4

Bibliothèque nationale du Canada
Direction du développement des collections

Service des thèses canadiennes
sur microfiche

## NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

## AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED

LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE

Canada

NL-338 (r. 85/09)

National Library of Canada | Bibliothèque nationale du Canada

Ottawa, Canada
K1A 0N4

TC –

ISBN 0-315-21257-6

CANADIAN THESES ON MICROFICHE SERVICE – SERVICE DES THÈSES CANADIENNES SUR MICROFICHE

## PERMISION TO MICROFILM – AUTORISATION DE MICROFILMER

• Please print or type – Écrire en lettres moulées ou dactylographier

### AUTHOR – AUTEUR

Full Name of Author – Nom complet de l'auteur

Stephen Daniel Kmet

| Date of Birth – Date de naissance | Canadian Citizen – Citoyen canadien |
|---|---|
| Nov 5 ~~1985~~ 1959 | [X] Yes / Oui [ ] No / Non |
| **Country of Birth – Lieu de naissance** | **Permanent Address – Résidence fixe** |
| Canada | General Delivery<br>Leslieville Alta.<br>T0M-1H0 |

### THESIS – THÈSE

Title of Thesis – Titre de la thèse

Some Isomorphism Invariants of ZG

| Degree for which thesis was presented<br>Grade pour lequel cette thèse fut présentée | Year this degree conferred<br>Année d'obtention de ce grade |
|---|---|
| Master of Science | 1985 |
| **University – Université** | **Name of Supervisor – Nom du directeur de thèse** |
| University Alberta | Dr Sehgal |

### AUTHORIZATION – AUTORISATION

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to **microfilm** this thesis and to **lend** or **sell copies of the film.**

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈQUE NATIONALE DU CANADA de **microfilmer** cette thèse et de **prêter** ou de **vendre des exemplaires du film.**

L'auteur se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.

ATTACH FORM TO THESIS – VEUILLEZ JOINDRE CE FORMULAIRE À LA THÈSE

| Signature | Date |
|---|---|
| Stephen D. Kmet | April 20th 1985 |

NL-91 (r. 84/03)

Canada

THE UNIVERSITY OF ALBERTA

# SOME ISOMORPHISM INVARIANTS OF $\mathbb{Z}G$

by

STEPHEN D. KMET

(C)

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

SPRING, 1985

THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR Stephen D. Kmet

TITLE OF THESIS Some Isomorphism Invariants of $\mathbb{Z}G$

DEGREE FOR WHICH THESIS WAS PRESENTED Master of Science

YEAR THIS DEGREE GRANTED 1985

Permission is hereby granted to THE UNIVERSITY OF ALBERTA LIBRARY to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

(Signed) Stephen D. Kmet

PERMANENT ADDRESS:
General Delivery
Lesliesville Alberta
T0M-1H0

DATED April 19th 1985

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled .....

SOME ISOMORPHISM INVARIANTS OF ZG

.....

submitted by ...STEPHEN D. KMET.....

in partial fulfilment of the requirements for the degree of Master of Science.

.....S.K. Sehgal.....
Supervisor

.....Gerald Cliff.....

.....A. Weiss.....

.....Alan Adam.....

Date April 18, 85.....

## ABSTRACT

Let $\mathbb{Z}G$ and $\mathbb{Z}H$ be group rings of the groups $G$ and $H$ over the ring of integers $\mathbb{Z}$. Suppose the torsion elements, $T(G)$ and $T(H)$ of $G$ and $H$, form subgroups. When $\mathbb{Z}G$ is isomorphic to $\mathbb{Z}H$ as $\mathbb{Z}$ algebras, we then can find factor groups of $G$ and $H$ that are isomorphic.

In this thesis we show for arbitrary groups that,

(1) $D_i(G) \cap T(G)/D_{i+j}(G) \cap T(G) \simeq D_i(H) \cap T(H)/D_{i+j}(H) \cap T(H)$

for $1 \leq j \leq i + 2$,

(2) $\gamma_i(T(G))/\gamma_{i+j}(T(G)) \simeq \gamma_i(T(H))/\gamma_{i+j}(T(H))$

for $1 \leq j \leq i$,

(3) $\delta_i(T(G))/\delta_2([G,\delta_i(T(G))]) \simeq \delta_i(T(H))/\delta_2([H,\delta_i(T(H))])$

where $D_i(G)$, $\gamma_i(G)$ and $\delta_i(G)$ are the $i^{th}$ subgroups of the dimension, lower central, and derived series, respectively. Furthermore when $G$ is torsion we have,

$$\gamma_i(G)/\gamma_{i+j}(G) \simeq \gamma_i(H)/\gamma_{i+j}(H) \quad \text{for} \quad 1 \leq j \leq i + 2.$$

ACKNOWLEDGEMENT

I wish to express my gratitude to Dr. Sehgal for his advice and guidance in these past years. And with that, a special thank you to Sudarshan for those happy gatherings with his family and friends. Also, I thank Brian for showing what dedication, determination and companionship mean.

Finally, I wish all the best for my family and special friends Bart, Byron, Ron, Scott, Ron, John, Ken and Elaine. They made university my home away from home.

TABLE OF CONTENTS


| CHAPTER | | PAGE |
|---|---|---|
| I | INTRODUCTION . . . . . | 1 |
| II | PRELIMINARIES . . . . . | 4 |
| | 2.1 Groups . . . . . | 4 |
| | 2.2 Group Rings . . . . . | 9 |
| III | TORSION UNITS . . . . . | 14 |
| | 3.1 Introduction . . . . . | 14 |
| | 3.2 Torsion Units of $1 + \Delta(G,A)\Delta(N)$ . . . . . | 15 |
| IV | ISOMORPHISM INVARIANT IDEALS AND FACTOR GROUPS . . . | 23 |
| | 4.1 Introduction . . . . . | 23 |
| | 4.2 Invariant Ideals . . . . . | 24 |
| | 4.3 Invariant Factor Groups . . . . . | 27 |
| | REFERENCES . . . . . | 34 |

# CHAPTER I

## INTRODUCTION

If $R$ is a ring and $G$ is a group, we can define an $R$-algebra $RG$, in a natural way. This $R$-algebra, called a group ring, dates to the work of Frobenius. He used $RG$ as an aid in studying representation theory. Since that time, considerable interest has been shown in the study of group rings as a topic in its own right.

Passman [13] and Dade [6] showed that the ring $R$ affects the properties of $RG$ considerably. Thus, research on $RG$ has been on studying $RG$ for rings $R$, with certain well defined properties. In this thesis we will look at the ring $R = \mathbb{Z}$, the ring of integers.

For the group ring $\mathbb{Z}G$, Passman proved that unit elements of finite order that can be expressed as:

$$s + \sum_{g \neq 1} r_g g \quad \text{with} \quad s, r_g \in \mathbb{Z},\ g \in G \quad s \neq 0$$

are in fact equal to $s$. Based on this result, many authors (see [19]) have been able to determine some of the structure and properties of $\mathbb{Z}G$. Because of the numerous results obtained for the group ring $\mathbb{Z}G$, it was proposed that the properties of $\mathbb{Z}G$ are completely determined by $G$. That is:

If $\mathbb{Z}G$ is isomorphic to $\mathbb{Z}H$ as a $\mathbb{Z}$-algebra, then is $G$ isomorphic to $H$?

Affirmative results have been given for abelian, finite metabelian groups

of finitely generated nilpotent class two groups, circle groups of nilpotent class three, torsion free nilpotent groups, and many others. (See [4, 14, 17, 11, 3]). Because of these affirmative results and that no counterexamples have been found to date, it is conjectured that the answer to the previous question is yes.

An approach to proving this conjecture, is to find normal subgroups $A$ and $B$ of $G$, and $N$ and $M$ of $H$, with the factor groups $A/B$ and $N/M$, isomorphic. By trying to show this for large subgroups $A$, and small subgroups $B$, we can obtain partial answers to the question. Our thesis will be concerned with finding some of these factor groups.

Chapter two defines our notation, and states definitions and known theorems and lemmas on groups and group rings, that we will need. We also define a new central series $\{T_n(G)\}$ of a group $G$. In later chapters we show that many of its factor groups, $T_i(G)/T_{i+j}(G)$ are invariants.

In chapter three, we look at units of the ideal,

$1 + \Delta(G,A)\Delta(G,Z)$ with $A$ a torsion nilpotent subgroup of $G$, and $Z$ the center of $A$.

We show the torsion units are all equal to 1, extending a well known result of [4]:

If $G$ is a torsion group with $A$ an abelian normal subgroup of $G$, then the torsion units of $1 + \Delta^2(G,A)$ are trivial.

We use this result in proving our main theorem, but note that it is interesting on its own.

Finally, in chapter four, we prove our main result. We show that:

Let $G$ and $H$ be groups with the torsion elements, $T(G)$ and $T(H)$, of $G$ and $H$ respectively, forming subgroups. If $\mathbb{Z}G$ is isomorphic to $\mathbb{Z}H$ as a $\mathbb{Z}$ algebra, then we have the following factor groups of $G$ isomorphic to their counterparts in $H$, for all $i$.

| | | |
|---|---|---|
| (1) | $T_i(G)/T_{i+j}(G)$ | for $1 \leq j \leq i+2$. |
| (2) | $D_i(G) \cap T(G)/D_{i+j}(G) \cap T(G)$ | for $1 \leq j \leq i+2$, $D_i(G)$ the $i^{th}$ dimension subgroup. |
| (3) | $\gamma_i(T(G))/\gamma_{i+j}(T(G))$ | for $1 \leq j \leq i$, $\gamma_i$ the $i^{th}$ member of the lower central series. |
| (4) | $\delta_i(T(G))/\delta_{i+1}(T(G))$ | $\delta_i$ the $i^{th}$ member of the derived series. |
| (5) | $\delta_i(T(G))/[G,\delta_i(T(G))]'$ | where $G' = \delta_2(G)$. |
| (6) | $\gamma_i(G)/\gamma_{i+j}(G)$ | for $1 \leq j \leq i+2$ and $G$ and $H$ torsion. |

This result extends the results of the authors Furukawa [9], Whitcomb [20] and Ritter-Sehgal [15]. We note that these authors each used a distinct method of proof to obtain their results.

# CHAPTER II

# PRELIMINARIES

In this chapter we establish our notation and some preliminary lemmas.

## Section 2.1. GROUPS

Throughout this thesis we will let $G$ and $H$ represent groups. $A$, $B$, $N$ and $M$ will represent subgroups of $G$ or $H$.

Definition 2.1.1. A group $G$ is said to be *nilpotent* if the $i^{th}$ element, $\gamma_i(G)$, of the *lower central series* equals 1 for some $i$. The lower central series $\{\gamma_i(G)\}$, is defined as follows:

(i) $\gamma_1(G) = G$

(ii) $\gamma_{i+1}(G) = [G,\gamma_i(G)]$ for $i \geq 1$

where $[G,H]$ is the subgroup generated by all the elements $[g,h] = g^{-1}h^{-1}gh$, with $g \in G$ and $h \in H$.

Definition 2.1.2. The *derived series*, $\{\delta_i(G)\}$, is inductively defined as,

(i) $\delta_1(G) = G$

(ii) $\delta_{i+1}(G) = [\delta_i(G),\delta_i(G)]$ for $i \geq 1$.

$\delta_2(G)$ is sometimes written as $G'$.

We will need the following results about *bounded exponent* nilpotent groups. $G$ is of bounded exponent if there is an $n \in \mathbb{Z}$, $n \neq 0$,

$\mathbb{Z}$ the ring of integers, with $g^n = 1$ for every $g \in G$.

Lemma 2.1.3 [16]. Let $G$ be a nilpotent group of bounded exponent. Then

$$G = A_p \times A_{p'},$$

where $A_p$ is the p-Sylow subgroup, and $A_{p'}$ is a normal p'-subgroup ($a \in A_{p'}$, with $a^{p^n} = 1$ implies $a = 1$).

An element $g$ of $G$ is said to be *torsion* if $g^n = 1$ for some natural number $n$. A group $G$ is said to be torsion if every element of $g$ is torsion. $G$ is called torsion free if it contains no torsion elements.

Lemma 2.1.4 [16, p. 53]. Let $N$ and $M$ be subsets of a torsion nilpotent group $G$. Suppose $N$ and $M$ are both finite. Then the subgroup generated by $N$ and $M$, $\langle N,M \rangle$, is a finite nilpotent subgroup.

An element $g$ of $G$ is *central* if for all $g_1 \in G$ we have $g_1 g = g g_1$. $G$ is said to be *abelian* if every element of $G$ is central. We will need the following.

Lemma 2.1.5 [7]. Let $G$ be an abelian group of bounded exponent. Then $G$ is a direct sum of cyclic groups.

We will need to use the three subgroup lemma.

Lemma 2.1.6 [16, p. 44]. Let $M,N,B$ be subgroups of $G$. Consider

the following three subgroups of G.

(i) [[M,N],B]

(ii) [[N,B],M]

(iii) [[B,M],N]

If any two of these subgroups are contained in a normal subgroup of G, then so is the third.

We now define a set of subgroups of a group G. In later chapters we shall need the properties of these subgroups.

Definition 2.1.7. Let G be a group with T(G), the set of torsion elements of G, a subgroup in G. We inductively define the subgroups $T_i(G)$ as

$$T_1(G) = T(G) \quad \text{and} \quad T_{i+1}(G) = [G, T_i(G)].$$

Lemma 2.1.8. Let G be a group with T(G) a normal subgroup of G. Then

(i) $T_i(G)$ is normal in G for all i,

(ii) $T_i(G)/T_{i+1}(G)$ is central in the factor group $G/T_{i+1}(G)$ for all i,

(iii) $[T_i(G), T_j(G)] \leq T_{i+j}(G)$ for all $i \geq 1$ and $j \geq 1$.

PROOF: (i) We prove (i) by using induction on i. For the case i = 1, we use that $T_1(G) = T(G)$ is normal by the hypothesis.

Note that $T_{i+1}(G)$ is generated by the elements [g,t] with $g \in G$ and $t \in T_i(G)$. Using the identity,

$$g[g_1,t_1][g_2,t_2] \cdots [g_n,t_n]g^{-1} = g[g_1,t_1]g^{-1} \cdot g[g_2,t_2] \cdots g[g_n,t_n]g^{-1},$$

it is sufficient to show that,

$$g[g_1,t_1]g^{-1} \in T_{i+1}(G),$$

for $t_1 \in T_i(G)$ and $g, g_1 \in G$. But,

$$\begin{aligned} g[g_1,t_1]g^{-1} &= gg_1^{-1}t_1^{-1}g_1t_1g \\ &= (gg_1^{-1}g^{-1}) \cdot (gt_1^{-1}g^{-1}) \cdot (gg_1g^{-1}) \cdot (gt_1g^{-1}) \\ &= [gg_1g^{-1}, gt_1g^{-1}]. \end{aligned}$$

By the induction hypothesis and the definition of $T_{i+1}(G)$ we conclude that $T_{i+1}(G)$ is normal in $G$.

(ii) Note that $[T_i(G),G] = T_{i+1}(G)$ for all $i$. Thus

$$[T_i(G),G]/T_{i+1}(G) = 1.$$

Therefore we see that $T_i(G)/T_{i+1}(G)$ is central in $G/T_{i+1}(G)$ for all $i$.

(iii) We use induction on $i$ to show (iii). For the case $i = 1$ we observe that,

$$[T(G),T_j(G)] \subseteq [G,T_j(G)], \quad \text{for all } j \geq 1.$$

By the definition of $T_j(G)$ we have

$$[G,T_j(G)] = T_{j+1}(G).$$

Suppose the hypothesis is true for $i$. We first show that the two subgroups,

$$[[T_i(G),T_j(G)],G] \quad \text{and} \quad [[T_j(G),G],T_i(G)]$$

are contained in $T_{i+j+1}(G)$ for all $j$.

By the induction hypothesis,

$$[[T_i(G),T_j(G)],G] \subseteq [T_{i+j}(G),G].$$

Therefore, by using the definition of $T_{i+j+1}(G)$, we have that

$$[[T_i(G),T_j(G)],G] \subseteq T_{i+j+1}(G).$$

By the definition of $T_{j+1}(G)$ we have that

$$[[T_j(G),G],T_i(G)] = [T_{j+1}(G),T_i(G)].$$

Using the identity $g[g,h]g^{-1} = [h,g^{-1}]$, and the normality of $T_i(G)$ we see that,

$$[T_{j+1}(G),T_i(G)] = [T_i(G),T_{j+1}(G)].$$

By the induction hypothesis,

$$[T_i(G),T_{j+1}(G)] \subseteq T_{i+j+1}(G).$$

Applying lemma 2.1.6 we conclude that

$$[[G,T_i(G)],T_j(G)] \subseteq T_{i+1+j}(G), \quad \text{for all} \quad j.$$

But

$$[[G,T_i(G)],T_j(G)] = [T_{i+1}(G),T_j(G)].$$

□

Finally, we remark that we will use the notation $\bar{g}$ and $\bar{N}$ to represent elements and subgroups of a factor group $G/M$, when it is clear which subgroup $M$ of $G$ we are using.

## Section 2.2. GROUP RINGS

We formally state the definition of a group ring.

Definition 2.2.1. Let $G$ be a group and $R$ a commutative ring with identity. The group ring $RG$ is defined as the set of all formal sums $\{\Sigma r_g g \mid r_g \in R,\ g \in G$ and almost all $r_g = 0\}$ with equality defined as,

$$\sum r_g g = \sum s_g g \text{ if and only if } r_g = s_g \text{ for all } g \in G.$$

$RG$ is given ring properties with the two operators '+', and '·' defined as:

$$\sum r_g g + \sum s_g g = \sum (r_g + s_g)g \text{ and}$$

$$(\sum r_g g) \cdot (\sum s_g g) = \sum_g (\sum_h r_{gh^{-1}} s_h)g .$$

$RG$ is a ring with $1 \cdot e$ and $\Sigma\, 0 \cdot g$ as its multiplicative and additive identities respectively, where $1$ and $0$ are the ring $R$'s multiplicative and additive identities, and $e \in G$ is the group identity element. We write $1 \cdot e$ as $1$, and $\sum_{g \in G} 0g$ as $0$.

This thesis will be concerned with the group ring $\mathbb{Z}G$, $\mathbb{Z}$ the

ring of integers. The following theorem by Passman [13] is the foundation of our results.

Theorem 2.2.2. Let $\mathbb{Z}$ be the ring of integers. Suppose $\delta$ is a torsion unit of the group ring $\mathbb{Z}G$, with $\delta = \Sigma\, \delta_g g$, $\delta_g \in \mathbb{Z}$ and $g \in G$. Then, if $\delta_1 \neq 0$ we have that,

$$\delta = \pm 1.$$

We denote the set of units of a subset $I$ of $RG$ as $U(I)$. We write the set of torsion units as $TU(I)$.

Certain ideals of $\mathbb{Z}G$ are of special interest to us. If $N$ is a normal subgroup of $G$, we can then define a natural homomorphism $\theta$, to the group ring $\mathbb{Z}(G/N)$. Where $\theta : \mathbb{Z}G \to \mathbb{Z}(G/N)$ is defined as

$$\theta(\sum z_g g) = \sum z_g \overline{g}.$$

We are interested in the kernel of such maps which we denote by $\Delta(G,N)$. The following is an equivalent definition of $\Delta(G,N)$.

Definition 2.2.3. Let $N$ be a normal subgroup of $G$. Then the two sided ideal, $\Delta(G,N)$ is defined as the ideal generated by the set $\{n-1 \mid n \in N\}$.

We write $\Delta(G,G)$ as $\Delta(G)$, and $\Delta(G,A)\Delta(G,A)$ as $\Delta^2(G,A)$.

Ideals of the form $\Delta(G,N)$ are called *augumentation ideals*. An important corollary of theorem 2.2.2 is the following.

Corollary 2.2.4. Let $N$ be a torsion central subgroup of $G$. Then

$$TU(1 + \Delta(G,N)) = N.$$

PROOF: Let $\delta \in 1 + \Delta(G,N)$ be a torsion unit. Then

$$\delta - 1 = \sum_{i=1}^{n} x_i \delta_{x_i},$$

where $\delta_{x_i} \in \Delta(N)$, and $x_i$ are coset representatives of $N$ in $G$. We see that $1 + \delta_e = \Sigma z_i n_i$, $z_i \in \mathbb{Z}$ and $n_i \in N$, does not equal 0. Choose $n_i$ in the support of $1 + \delta_e$ with $z_i \neq 0$. Because $N$ is torsion central, we have that $n_i^{-1}\delta$ is a torsion unit. Furthermore if $n_i^{-1}\delta = \Sigma z_g g$, we have $z_e \neq 0$. Applying theorem 2.2.2. we conclude that $\delta = n_i$. □

If $I$ is an ideal of $RG$ then, $N = G \cap (1 + I)$ is a normal subgroup of $G$. A series of subgroups that are formed in this fashion are the *dimension subgroups*.

Definition 2.2.5. Let $RG$ be a group ring. We define the $i^{th}$ dimension subgroup, $D_{i,R}(G)$ as,

$$D_{i,R}(G) = G \cap (1 + \Delta(G)^i), \text{ for } i \geq 1.$$

When it is understood which ring $R$ is being referred to, we write $D_{i,R}(G)$ as $D_i(G)$. We will use some of the following well known properties [12, p. 21] of $D_i(G)$.

Lemma 2.2.6. Let $RG$ be a group ring. Then

(i) $D_i(G)$ is normal in $G$ for all $i \geq 1$,

(ii) $[D_i(G), D_j(G)] \subseteq D_{i+j}(G)$ for all $i \geq 1$ and $j \geq 1$.

(iii) $D_i(G)/D_{i+1}(G)$ is central in $G/D_{i+1}(G)$ for all $i \geq 1$.

The following is a useful theorem concerning the ideal $\Delta(G)\Delta(G,N)$.

Theorem 2.2.7. Let $N$ be a normal subgroup of $G$. Then

$$G \cap (1 + \Delta(G)\Delta(G,N)) = \gamma_2(N).$$

Note that if $N$ is abelian, then the above intersection is 1.

We present some well known identities of $\mathbb{Z}G$.

Lemma 2.2.8. In the group ring $\mathbb{Z}G$, we have that

(i) $(g-1) + (h-1) = (gh-1) - (g-1)(h-1)$ and

(ii) $(g-1) + (g^{-1}-1) = -(g-1)(g^{-1}-1)$.

An immediate result is the following.

Lemma 2.2.9. In the group ring $\mathbb{Z}G$, we have that

(i) For any $\delta \in \Delta(G,A)$, $\delta$ can be written as

$\delta = c(a-1) + \beta$, where $c \in \mathbb{Z}$ and $\beta \in \Delta(G)\Delta(G,A)$, $a \in A$.

(ii) If $\delta = c(a-1)$ then $\delta$ can be written as,

$$\delta \equiv (a^c - 1) \bmod \Delta(\langle a \rangle)^2.$$

Finally, we define our notation for isomorphisms. In this

thesis all isomorphisms, $\theta: RG \to RH$ will be R-algebra isomorphisms. That is,

$$\theta(r\delta) = r\theta(\delta) \quad \text{for all} \quad \delta \in RG \quad \text{and}$$

$$\theta(x+y) = \theta(x) + \theta(y), \quad \theta(xy) = \theta(x)\theta(y) \quad \text{for all} \quad x, y \in RG.$$

We write $RG$ is isomorphic to $RH$, as $RG \underline{\sim} RH$. If the isomorphism is of importance we write $RG$ is isomorphic to $RH$ by the isomorphism $\theta$, as $RG \overset{\theta}{\underline{\sim}} RH$.

# CHAPTER III

# TORSION UNITS

## Section 3.1. INTRODUCTION

In this chapter, we concern ourselves with the following question:

Question 3.1.1. Under what conditions do the torsion units of the ideals

(i) $1 + \Delta(G,N)$ and,

(ii) $1 + \Delta(G,A)\Delta(G,N)$

equal $T(N)$ and $1$ respectively.

We note that any solution to (ii) with $A = G$, requires $T(N)$ to be abelian, since $\gamma_2(T(N))$ is contained in $1 + \Delta(G)\Delta(N)$. We believe the normal subgroups $N$, with $T(N)$ abelian, form the solution class of (ii).

Solutions to (i) have been based on theorem 2.2.2. An immediate corollary to this result is that torsion central subgroups of $G$ satisfy question 3.1.1 (i). Furukawa [8] added to the solution class the set of nilpotent groups $G$, with $N$ central. We extend these results by showing that any central subgroup $N$ of $G$ satisfies (i).

Investigation into (ii) has been restricted to the ideals,

(a) $1 + \Delta(G,A)^2$, with $A$ abelian

(b) $1 + \Delta(G)\Delta(G,A)$, with $A$ abelian.

It is well known [4] that if $A$ is a normal abelian subgroup of a finite group $G$, then (ii) is satisfied for the ideal in (a). We extend this result by showing that if $A$ is a nilpotent normal torsion subgroup of a group $G$, with $Z$ the center of $A$, then the ideal $1 + \Delta(G,A)\Delta(G,Z)$ satisfies (ii).

Cliff-Sehgal-Weiss [4] have shown (b) satisfies (ii) for finite metabelian groups $G$, with $A$ abelian, $G/A$ abelian. Passman-Smith [14] showed that (b) satisfies (ii) for torsion nilpotent groups $G$, with $A$ abelian. We extend Passman-Smith's results to groups $G$, with $T_n(G) = 1$.

## Section 3.2. TORSION UNITS OF $1 + \Delta(G,A)\Delta(N)$

We need the following result due to Hartley [10].

Theorem 3.2.1 [10]. Let $G$ be a nilpotent p-group of bounded exponent. If $R$ is a ring with identity satisfying $\bigcap_{n\geq 1} p^n R = 0$ then $\bigcap_{n\geq 1} \Delta_R(G)^n = 0$. $\bigcap_{n\geq 1} \Delta_R(G)^n$ is written as $\Delta_R^{\omega}(G)$.

Lemma 3.2.2 [19, p. 35]. Let $N$ be a normal subgroup of $G$. If $R$ is a commutative ring with identity, then

$$\bigcap_{n\geq 1} \Delta(G,N)^n = \left(\bigcap_{n\geq 1} \Delta(N)^n\right)RG$$

Corollary 3.2.3. Let $N$ be a normal nilpotent p-subgroup of bounded exponent in a group $G$. If $R$ is a commutative ring with identity satisfying $\bigcap_{n\geq 1} p^n R = 0$ then,

$$\Delta^{\omega}(G,N) = 0.$$

We begin with,

Theorem 3.2.4. Let $A$ be a nilpotent p-group of bounded exponent, where $p$ is a fixed prime. Let $Z$ be a central subgroup of $A$, and $I$ an ideal in $\mathbb{Z}A$. Then

$$I \subseteq \Delta(A)\Delta(Z) \text{ and } pI \subseteq I^p \text{ implies } I = 0.$$

PROOF: Let us first suppose $Z$ is finite. In this case the result is proved by induction on the order of $Z$. If $|Z| = 1$ then $\Delta(A)\Delta(Z) = 0$. Hence $I = 0$.

If $|Z| = p^n$, choose an element $z \in Z$ of order $p$. Applying the induction hypothesis to $\mathbb{Z}(A/\langle z\rangle)$ we conclude $\overline{I} = 0$. Thus,

$$I \subseteq \Delta(A,\langle z\rangle) = (1 - z)\mathbb{Z}A.$$

We claim that

$$(*) \qquad (1 - z)\mathbb{Z}A \cap \Delta(A)\Delta(Z) = (1 - z)\Delta(A).$$

To see this, let $\alpha$ be an element in the intersection. Write

$$\alpha = (1 - z)\gamma, \quad \gamma \in \mathbb{Z}A.$$

By using lemma 2.2.9 we have

$$\alpha = c(1 - z) + \delta, \text{ for } c \in \mathbb{Z}, \ \delta \in (1 - z)\Delta(A), \text{ and}$$

$$\alpha = (1 - z^c) + \delta_1 \text{ for } \delta_1 \in (1 - z)\Delta(A).$$

But $\alpha, \delta_1 \in \Delta(A)\Delta(Z)$, and thus

$$1 - z^c \in \Delta(A)\Delta(Z).$$

We conclude by theorem 2.2.7 that

$$z^c \in \gamma_2(Z)$$

and since $Z$ is abelian, we see

$$1 - z^c = 0.$$

We have shown $\alpha \in (1 - z)\Delta(A)$, establishing (*). It follows from (*) that,

$$I \subseteq \Delta(A)\Delta(Z) \cap (1 - z)\mathbb{Z}A = (1 - z)\Delta(A).$$

Suppose that $I \subseteq (1 - z)\Delta(A)^{\ell}$ where $\ell$ is a natural number. By hypothesis

$$pI \subseteq I^p \subseteq (1 - z)^p\Delta(A)^{p\ell}.$$

Since $z$ has order $p$ we have that

$$1 = (1 - z + z)^p = 1 + \binom{p}{1}(1 - z)z^{p-1} + \ldots + (1 - z)^p,$$

implying $(1 - z)^p \in p(1 - z)\mathbb{Z}A$. Hence

$$pI \subseteq p(1 - z)\Delta(A)^{p\ell}.$$

Thus $I \subseteq (1 - z)\Delta(A)^{p\ell}$ implying $I \subseteq \Delta(A)^{\omega}$. By theorem 3.2.1, $I = 0$. This completes the theorem for $Z$ a finite subgroup.

Now suppose $Z$ is infinite. By lemma 2.1.5, $Z$ can be expressed

as a direct sum of cyclic groups. Let $B_v$ be a subgroup of $Z$ obtained by dropping a finite number of factors. Applying the finite case of this theorem to $A/B_v$, we conclude that $I \subseteq \Delta(G,B_v)$. But $\cap_v \Delta(G,B_v) = 0$, showing $I = 0$. □

Lemma 3.2.5. Let $A$ be a normal nilpotent subgroup of exponent $n$ contained in $G$. Then $U(1 + \Delta(G,A))$ has no torsion elements of order relatively prime to $n$.

PROOF: Suppose $U(1 + \Delta(G,A))$ has a torsion element $(1 + \delta)$ with $(1 + \delta)^q = 1$, and $q$ is a prime not dividing $n$. It is enough to show $\delta = 0$.

We proceed by induction on the number of prime factors of $n$. If $n$ has more than one distinct prime factor, by applying lemma 2.1.3, we can write $A = A_1 \times A_2$, where $A_1$ is a p-Sylow subgroup and $A_2$ is a p'-group, where $p|n$ and $p$ and $q$ are relatively prime. By factoring $A_1$, and applying the induction hypothesis, we conclude $\delta \in \Delta(G,A_1)$. Thus it suffices to show for $A$, a bounded exponent p-group with $(1 + \delta)^q = 1$, that $\delta = 0$.

We have,

$$1 = (1 + \delta)^q = 1 + q\delta + \binom{q}{2}\delta^2 + \ldots + \delta^q$$

and thus $q\delta \in \Delta(G,A)^2$. Using the above identity with $\delta = a - 1$ we have

$$p^m(1 - a) \in \Delta(G,A)^2, \quad \text{where } a^{p^m} = 1.$$

Therefore there exists an $m$, with

$$p^m\delta \in \Delta(G,A)^2.$$

Since there exists $c, b \in \mathbb{Z}$ with $cp^m + bq = 1$, we have $\delta \in \Delta(G,A)^2$. Repeating this argument, we see that $\delta \in \Delta(G,A)^\omega$. Applying corollary 3.2.3 we have $\delta = 0$.

□

We now present new solutions for Question 3.1.1 (ii).

Theorem 3.2.6. Let $A$ be a nilpotent normal torsion subgroup of a group $G$. Let $Z$ be the centre of $A$. Then

$$TU(1 + \Delta(A)\Delta(Z)\mathbb{Z}G) = 1.$$

PROOF: Note $Z$ is normal in $G$, since it is a characteristic subgroup of $A$. Suppose $(1+\delta)^p = 1$, where $p$ is a prime and $\delta = \sum_1^n x_i\delta_i \in \Delta(A)\Delta(Z)\mathbb{Z}G$, with $\delta_i \in \Delta(A)\Delta(Z)$ and the $x_i$ are different coset representatives of $G/A$. Let

$$W = \{a \in A \mid m_a \neq 0,\ \delta_i = \Sigma\, m_a a \text{ for some } i,\ m_a \in \mathbb{Z}\}.$$

$$Y = \{a_i, b_i \mid \delta = \sum_{i=1}^n x_i d_i (a_i - 1)(b_i - 1),\ d_i \in \mathbb{Z}\ a_i \in A\ b_i \in Z\}.$$

Since $W$ and $Y$ are both finite, by lemma 2.1.4 we conclude that $\langle W,Y\rangle$ is a nilpotent group of bounded exponent. By replacing $A$ with the normal subgroup generated by $\langle W,Y\rangle$, we may assume $A$ is a normal nilpotent subgroup of bounded exponent in $G$.

We use induction on the number of primes in the exponent. By lemma 2.1.3 we can write $A = A_1 \times B$, where $A_1$ is a p-group and $B$ is a p'-group. Factoring by $A_1$ and applying lemma 3.2.5 we have

that $\delta \in \Delta(G,A_1)$.

By replacing $A$ by $A_1$, we may assume $A$ is also a p-group. Let $I$ be the smallest ideal of $\mathbb{Z}A$ containing $\delta_1,\ldots,\delta_n$ and invariant under conjugation by $G$. Since $\delta_i \in \Delta(A)\Delta(Z)$ and $\Delta(A)\Delta(Z)$ is invariant under conjugation by $G$, we have $I \subseteq \Delta(A)\Delta(Z)$. We claim $pI \subseteq I^p$.

The equality $(1+\delta)^p = 1$ implies

$$p\delta + \binom{p}{2}\delta^2 + \ldots + \delta^p = 0.$$

Therefore $p\delta \in p\delta^2\,\mathbb{Z}G + I^p$. Hence

$$p\delta \in p\delta^p\mathbb{Z}G + I^p \subseteq I^p\mathbb{Z}G$$

or,

$$\sum_{i=1}^{n} px_i\delta_i \in I^p\mathbb{Z}G .$$

Therefore $p\delta_i \in I^p$ and $pI \subseteq I^p$. By theorem 3.2.4 we know $I = 0$, and thus $\delta = 0$.

□

To obtain our second solution to Question 3.2.1 (ii) we need a result of Sehgal's.

Theorem 3.2.7 [18]. Let $\mathbb{Z}G$ be a group ring. Then $T(U(\mathbb{Z}G)) = \pm 1$ if and only if $G$ is torsion free.

Theorem 3.2.8. Let $N$ be a central subgroup of $G$. Then

(i) $T(U(1 + \Delta(G,N))) = T(N)$

(ii) $T(U(1 + \Delta(G)\Delta(N))) = 1.$

PROOF: (i) Assume $N$ is torsion free. Let $u$ be a torsion element of prime order $p$ with $u \in 1 + \Delta(G,N)$. Let $q$ be a prime, with $q \neq p,1$. By factoring $N$ by $N^{q^m} = \langle \{n^{q^m} | n \in N\} \rangle$ and applying corollary 2.2.4, we have that,

$$\bar{u} \in \bar{N}.$$

But $\bar{N}$ is a q-group and $\bar{u}$ an element of order $p$. Therefore, $\bar{u} = 1$ and

$$u - 1 \in \Delta(G,N^{q^m}) \quad \text{for all} \quad m.$$

This means that,

$$u - 1 \in \bigcap_{m \geq 1} \Delta(G,N^{q^m}).$$

By replacing $N$ with the normal subgroup generated by $\{n_i | u - 1 = \sum_{i=1}^{n} x_i a_i (n_i - 1)$, $n_i \in N$ $a_i \in \mathbb{Z}$, and $x_i$ coset representatives of $G/N\}$, we can assume $N$ is finitely generated. It is then easy to see that $\bigcap_{m \geq 1} (G,N^{q^m}) = 0$. Therefore $u = 1$.

If $T(N) \neq 1$, apply the above to $G/T(N)$. We can then conclude that $u \in 1 + \Delta(G,T(N))$. Applying corollary 2.2.4 we have that $u \in T(N)$.

(ii) Suppose $u$ is a torsion unit belonging to $1 + \Delta(G)\Delta(N)$. Since $\Delta(G)\Delta(N) \subseteq \Delta(G,N)$, by applying (i) to $u$, we conclude $u \in T(N)$. Thus

$$u \in T(N) \cap 1 + \Delta(G)\Delta(N),$$

and by theorem 2.2.7 we have $u = 1$. □

For groups with $T_n(G) = 1$, we now determine the torsion units of $1 + \Delta(G)\Delta(A)$, where $A$ is abelian.

Theorem 3.2.9. Let $T(G)$ be a subgroup of $G$ with $T_n(G) = 1$. If $A$ is a normal abelian subgroup of $G$, then

$$T(U(1 + \Delta(G)\Delta(A))) = 1$$

PROOF: We proceed by induction on $n$. For $n = 1$ we can apply theorem 3.2.7 to conclude $T(U(1 + \Delta(G)\Delta(A)) = 1$.

For $n > 1$ consider the factor group $G/T_{n-1}(G)$. By induction $TU(1 + \Delta(\overline{G})\Delta(\overline{A})) = 1$, and thus

$$T(U(1 + \Delta(G)\Delta(A))) \subseteq T(U(1 + \Delta(G, T_{n-1}(G)))).$$

Therefore by theorem 3.2.8, we have

$$T(U(1 + \Delta(G)\Delta(A))) \subseteq T_{n-1}(G).$$

By theorem 2.2.7 we conclude

$$T(U(1 + \Delta(G)\Delta(A))) = 1.$$

□

# CHAPTER IV

## ISOMORPHISM INVARIANT IDEALS AND FACTOR GROUPS

### Section 4.1. INTRODUCTION

In this chapter we consider the following questions:

Question 4.1.1. For which normal subgroups $N$ of $G$, does $\mathbb{Z}G \overset{\theta}{\simeq} \mathbb{Z}H$ imply the existence of a normal subgroup $M$ of $H$, with $\theta(\Delta(G,N)) = \Delta(H,M)$?

Question 4.1.2. For which normal subgroups $N$, $N_1$ of $G$, does $\mathbb{Z}G \overset{\theta}{\simeq} \mathbb{Z}H$ imply the existence of normal subgroups $M$, $M_1$ of $H$, with $N/N_1 \simeq M/M_1$?

A well known solution to question 4.1.1 is $\gamma_2(G)$. Whitcomb [20] showed $\theta\Delta(G,\gamma_2(G)) = \Delta(H,\gamma_2(H))$ by observing that the smallest ideal $I$ in $\mathbb{Z}G$ such that $\mathbb{Z}G/I$ is abelian is $\Delta(G,\gamma_2(G))$. Since this characterizes the ideal $I$, we have that $\theta\Delta(G,\gamma_2(G)) = \Delta(H,\gamma_2(H))$.

Later investigation [19, p. 100], showed for locally finite groups, that the lower, upper and derived series, were solutions to question 4.1.1.

Furukawa [9] extended these results to torsion groups and also showed that the dimension subgroups satisfied the question.

Using the solutions to question 4.1.1, each of these investigators were able to obtain solutions for question 4.1.2. Whitcomb showed $G/\delta_3(G) \simeq H/\delta_3(H)$ for $G$ locally finite. It was shown for locally finite groups that $\delta_i(G)/\delta_{i+1}(G) \simeq \delta_i(H)/\delta_{i+1}(H)$

and $\gamma_i(G)/\gamma_{i+1}(G) \simeq \gamma_i(H)/\gamma_{i+1}(H)$ for $i \geq 1$. Furukawa proved for a torsion group $G$, that $\gamma_i(G)/\gamma_{i+j}(G)$, $D_i(G)/D_{i+j}(G)$, and $\delta_i(G)/\delta_{i+1}(G)$ for $i \geq 1$ and $1 \leq j \leq i$, are isomorphic to their respective counterparts in $H$.

The approach taken by these authors was to use the correspondence of normal subgroups first introduced by Cohn and Livingstone [5] and Ziegenbalg [21]. This correspondence identified with each normal subgroup $N$ of $G$, the subgroup $\phi N = H \cap (1 + \theta(\Delta(G,N)))$ of $H$.

We note that not all solutions to question 4.1.1 have been dependent upon the results of question 4.1.2. By decomposing certain ideals of the group ring $\mathbb{Z}G$, Ritter-Sehgal [15] showed for groups $G$, with $G'$ of prime exponent, and $D_6(G) = 1$, that $\mathbb{Z}G \simeq \mathbb{Z}H$ implies $G' \simeq H'$.

In section 4.2, we use a new approach to construct ideals satisfying question 4.1.1. These solutions include the previous investigator's results, and allow us to extend all the previously mentioned results to question 4.1.2.

## Section 4.2. INVARIANT IDEALS

In this section we let $\theta$ denote an isomorphism between $\mathbb{Z}G$ and $\mathbb{Z}H$. $\theta$ is said to be *augmentation preserving* if for any $g \in G$, with $\theta(g) = \sum_{i=1}^{n} a_i h_i$, $a_i \in \mathbb{Z}$, $h_i \in H$, we have $\sum_{i=1}^{n} a_i = 1$. The following well known lemma allows us to assume that all isomorphisms are augmentation preserving.

Lemma 4.2.1 [19, p. 66]. If $\mathbb{Z}G \overset{\theta}{\sim} \mathbb{Z}H$, then there exists an isomorphism $\hat{\theta} : \mathbb{Z}G \to \mathbb{Z}H$ with $\hat{\theta}$ augmentation preserving.

We now construct some invariant ideals.

Lemma 4.2.2. Let $\mathbb{Z}G \overset{\theta}{\sim} \mathbb{Z}H$. Suppose A and B are normal torsion subgroups of G and H respectively with $\theta\Delta(G,A) = \Delta(H,B)$. Then $\theta\Delta(G,[G,A]) = \Delta(H,[H,B])$.

PROOF: We observe that

$$\Delta(G,[G,A]) \subseteq \Delta(G)\Delta(G,A) + \Delta(G,A)\Delta(G).$$

This follows because, for any $g \in G$, $a \in A$, we have that

$$g^{-1}a^{-1}ga - 1 = g^{-1}a^{-1}[(g-1)(a-1) - (a-1)(g-1)].$$

Let $a \in [G,A]$. Then

$$a - 1 \in \Delta(G)\Delta(G,A) + \Delta(G,A)\Delta(G).$$

Applying $\theta$ we obtain

$$\theta(a) \in 1 + \Delta(H)\Delta(H,B) + \Delta(H,B)\Delta(H).$$

Factoring by $[H,B]$ we conclude

$$\overline{\theta(a)} \in 1 + \Delta(\overline{H})\Delta(\overline{H},\overline{B}), \quad [\overline{H},\overline{B}] = 1.$$

It follows by theorem 3.2.8 (ii) that $\overline{\theta(a)} = 1$. Thus

$$\theta(a) \in 1 + \Delta(H,[H,B])$$

and

$$\theta\Delta(G,[G,A]) \subseteq \Delta(H,[H,B]).$$

The reverse inclusion follows by symmetry, proving the lemma. □

The next result is due to Furukawa [9].

Lemma 4.2.3. Suppose that $\mathbb{Z}G \overset{\theta}{\simeq} \mathbb{Z}H$. Suppose that $G_1$ and $H_1$ are subgroups of $G$ and $H$ respectively. Suppose $I$ is an ideal in $\mathbb{Z}G$ such that

$$\theta(G_1(1+I)) = H_1(1+\theta(I)).$$

Then

$$G_1/G_1 \cap (1+I) \simeq H_1/H_1 \cap (1+\theta(I)).$$

Lemma 4.2.4. Let $\mathbb{Z}G \overset{\theta}{\simeq} \mathbb{Z}H$. Suppose that $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$ are normal torsion subgroups of $G$ and $H$ respectively. Suppose that $\theta\Delta(G,A_i) = \Delta(H,B_i)$ for $i = 1,2$. Further, suppose that $A_2/A_1$ and $B_2/B_1$ are nilpotent. Then

$$\theta\Delta(G,[A_1,A_2]) = \Delta(H,[B_1,B_2]).$$

PROOF: From the identity, $a_1^{-1}a_2^{-1}a_1a_2 - 1 = a_1^{-1}a_2^{-1}[(a_1-1)(a_2-1) - (a_2-1)(a_1-1)]$, we conclude that

$$[A_1,A_2] \subseteq 1 + \Delta(G,A_1)\Delta(G,A_2) + \Delta(G,A_2)\Delta(G,A_1).$$

It follows for $a \in [A_1,A_2]$ that,

$$\theta(a) \in 1 + \Delta(H,B_1)\Delta(H,B_2) + \Delta(H,B_2)\Delta(H,B_1).$$

Factoring by $[B_1,B_2]$ we observe that

$$\overline{\theta(a)} \in 1 + \Delta(\overline{H},\overline{B}_1)\Delta(\overline{H},\overline{B}_2).$$

Since $B_2/B_1$ is nilpotent we have that $B_2/[B_1,B_2] = \overline{B}_2$ is also nilpotent. Applying theorem 3.2.6, we deduce that $\overline{\theta(a)} = 1$. Thus

$$\theta(a) \in 1 + \Delta(H,[B_1,B_2]).$$

The lemma follows by symmetry. □

Section 4.3. INVARIANT FACTOR GROUPS

We use the results of the previous section to prove our main theorem on invariant factor groups. We first need the following lemma:

Lemma 4.3.1. Let $\mathbb{Z}G \overset{\theta}{\cong} \mathbb{Z}H$. Suppose that $A \trianglerighteq N$ and $B \trianglerighteq M$ are normal subgroups of $G$ and $H$ respectively. Further, suppose that

(i) $A/N$, $B/M$ are torsion,

(ii) $[[A,G],[A,G]] \subseteq N$, $[[B,H],[B,H]] \subseteq M$,

(iii) $\theta\Delta(G,N) = \Delta(H,M)$, $\theta\Delta(G,A) = \Delta(H,B)$.

Then $A/N \cong B/M$.

PROOF: Write $G_1 = G/N$, $H_1 = H/M$. Then $\mathbb{Z}G_1 \cong \mathbb{Z}H_1$ with $A_1 = A/N$, $B_1 = B/M$, $\theta\Delta(G_1,A_1) = \Delta(H_1,B_1)$. Let $a_1 \in A_1$. Then $\theta(a_1) \in 1 + \Delta(H_1,B_1)$. Factoring by $[H_1,B_1]$ we have

$\overline{\theta(a_1)} \in 1 + \Delta(\overline{H}_1, \overline{B}_1)$. Since $\overline{B}_1$ is central in $\overline{H}_1$ it follows by corollary 2.2.4 that $\overline{\theta(a_1)} \in \overline{B}_1$. Thus

$$\theta(a_1) = b_0(1+\delta), \quad \delta \in \Delta(H_1, [H_1, B_1]), \quad b_0 \in B_1.$$

Using lemma 2.2.9 we can write

$$1 + \delta = b_2 + \gamma, \quad \gamma \in \Delta(H_1)\Delta([H_1, B_1]), \quad b_2 \in [H_1, B_1].$$

Thus $\theta(a_1) = b_1(1+\delta_1)$, $b_1 = b_0 b_2 \in B_1$, $\delta_1 \in \Delta(H_1)\Delta([H_1, B_1])$. We have seen that

$$\theta(A_1) \subseteq B_1(1 + \Delta(H_1)\Delta(H_1, [H_1, B_1])).$$

It follows by lemma 4.2.2 that $\theta\Delta(G_1, [G_1, A_1]) = \Delta(H_1, [H_1, B_1])$. Hence

$$\theta(A_1(1 + \Delta(G_1)\Delta(G_1, [G_1, A_1]))) \subseteq B_1(1 + \Delta(H_1)\Delta(H_1, [H_1, B_1])).$$

By symmetry we get equality. By lemma 4.2.3 we conclude that,

$$A_1/G_1 \cap (1 + \Delta(G_1)\Delta(G_1, [G_1, A_1])) \simeq B_1/B_1 \cap (1 + \Delta(H_1)\Delta(H_1, [H_1, B_1])).$$

It follows by (2) of the hypothesis and theorem 2.2.7 that $A_1 \simeq B_1$.

□

Now we present our main theorem.

Theorem 4.3.2. Suppose that $G$ and $H$ are groups such that the torsion elements, $T(G)$ and $T(H)$, of $G$ and $H$ respectively, form subgroups. Suppose $\mathbb{Z}G \simeq \mathbb{Z}H$. Then we have,

(1) $T_i(G)/T_{i+j}(T) \simeq T_i(H)/T_{i+j}(H)$ for $1 \leq j \leq i + 2$,

(2) $D_i(G) \cap T(G)/D_{i+j}(G) \cap T(G) \simeq D_i(H) \cap T(H)/D_{i+j}(H) \cap T(H)$

for $1 \leq j \leq i + 2$,

(3) $\gamma_i(T(G))/\gamma_{i+j}(T(G)) \simeq \gamma_i(T(H))/\gamma_{i+j}(T(H))$

for $1 \leq j \leq i$,

(4) $\delta_i(T(G))/\delta_{i+1}(T(G)) \simeq \delta_i(T(H))/\delta_{i+1}(T(H))$ for all $i$,

(5) $\delta_i(T(G))/[G,\delta_i(T(G))]' \simeq \delta_i(T(H))/[G,\delta_i(T(H))]'$

for all $i$ .

PROOF: By theorem 3.2.7 $\mathbb{Z}G/T(G) \simeq \mathbb{Z}G/\Delta(G,T(G))$ has no torsion units. Since $\Delta(G,T(G))$ is the smallest ideal $I$ such that $U(\mathbb{Z}G/I)$ is torsion free, we conclude that,

$$\theta\Delta(G,T(G)) = \Delta(H,T(H)).$$

It follows by lemma 4.2.2 that

$$\theta\Delta(G,T_i(G)) = \Delta(H,T_i(H)).$$

We shall apply lemma 4.3.1. Write

$$A = T_i(G), \quad N = T_{i+j}(G),$$

$$B = T_i(H), \quad M = T_{i+j}(H), \quad 1 \leq j \leq i + 2$$

By lemma 2.1.8 $[[T_i(G),G], [T_i(G),G]] \leq T_{2i+2}(G) \leq N$. The hypothesis of lemma 4.3.1 is satisfied. Thus $A/N \simeq B/M$.

(2) We wish to prove that $(D_i(G) \cap T(G))/(D_{i+j}(G) \cap T(G))$ with $1 \leq j \leq i + 2$, is an isomorphism invariant. We shall apply lemma 4.3.1. We shall first prove that

$$\theta\Delta(G,D_i(G) \cap T(G)) = \Delta(H,D_i(H) \cap T(H)).$$

We use induction on $i$. For $i = 1$, this says

$$\theta\Delta(G,T(G)) = \Delta(H,T(H))$$

which is true. Suppose $i \geq 1$. Since $D_{i+1}(G) \leq 1 + \Delta^{i+1}(G)$ and $D_{i+1}(G) \leq D_i(G)$ we have,

$$\theta\Delta(G,D_{i+1}(G) \cap T(G)) \leq \Delta(H,D_i(H) \cap T(H)) \quad \text{by induction}$$

and

$$\theta\Delta(G,D_{i+1}(G) \cap T(G)) \leq \Delta^{i+1}(H) \quad \text{by definition.}$$

Thus

$$(*) \qquad \theta\Delta(G,D_{i+1}(G) \cap T(G)) \leq \Delta(H,D_i(H) \cap T(H)) \cap \Delta^{i+1}(H).$$

Factoring by $D_{i+1}(H) \cap T(H)$ we conclude that

$$\overline{\theta(D_{i+1}(G) \cap T(G))} \leq 1 + \Delta(\overline{H},\overline{D_i(H) \cap T(H)}) \cap \Delta^{i+1}(\overline{H}).$$

But $\overline{D_i(H) \cap T(H)}$ is central in $\overline{H}$. It follows by corollary 2.2.4 that

$$\overline{\theta(D_{i+1}(G) \cap T(G))} \leq \overline{D_i(H) \cap T(H)}.$$

Thus if $g \in D_{i+1}(G) \cap T(G)$, then

$$\theta(g) = h(1 + \delta), \quad h \in D_i(H) \cap T(H), \quad \delta \in \Delta(H,D_{i+1}(H) \cap T(H)).$$

We know by (*) that $\theta(g) \in \Delta^{i+1}(H)$. Thus

$$h(1 + \delta) \in 1 + \Delta^{i+1}(H).$$

Hence it follows that $h \in D_{i+1}(H)$. We conclude that,

$$\theta\Delta(G, D_{i+1}(G) \cap T(G)) \subseteq \Delta(H, D_{i+1}(H) \cap T(H)).$$

Equality follows by symmetry. Applying lemma 4.3.1 with

$$A = D_i(G) \cap T(G), \quad N = D_{i+j}(G) \cap T(G), \quad 1 \leq j \leq i + 2$$

then $[[A,G],[A,G]] \subseteq (D_{i+1}(G) \cap T(G))' \subseteq D_{2i+2}(G) \cap T(G) \subseteq N$. Hence $D_i(G) \cap T(G)/D_{i+j}(G) \cap T(G)$ is preserved as desired.

(3) We shall prove that $\gamma_i(T(G))/\gamma_{i+j}(T(G))$, $1 \leq j \leq i$ are preserved. We know $\theta\Delta(G,(T(G))) = \Delta(H,T(H))$. Suppose that

$$\theta\Delta(G, \gamma_m(T(G))) = \Delta(H, \gamma_m(T(H))).$$

Applying lemma 4.2.4 and using the fact that $T(G)/\gamma_m(T(G))$ and $T(H)/\gamma_m(T(H))$ are nilpotent, we conclude that

$$\theta\Delta(G, \gamma_{m+1}(T(G)) = \Delta(H, \gamma_{m+1}(T(H))).$$

Notice that

$$[[\gamma_i(T(G)), G], [\gamma_i(T(G)), G]] \subseteq \gamma_{2i}(T(G)).$$

Similarly for H. If $A = \gamma_i(T(G))$, $B = \gamma_i(T(H))$, $N = \gamma_{2i}(T(G))$ and $M = \gamma_{2i}(T(H))$ then the hypotheses of lemma 4.3.1 are satisfied. The result follows.

(4) We wish to prove that $\delta_i(T(G))/\delta_{i+1}(T(G))$ is an isomorphism invariant. By lemma 4.2.4 we know that

$$\theta\Delta(G,\delta_i(T(G)) = \Delta(H,\delta_i(T(H))) .$$

Moreover, let $A = \delta_i(T(G))$ and $N = \delta_{i+1}(T(G))$. Then

$$[[A,G],[A,G]] \subseteq [\delta_i(T(G)),\delta_i(T(G))] \subseteq N.$$

It follows that $A/N$ is an isomorphism invariant.

(5) Let $A = \delta_i(T(G))$, $N = [G,\delta_i(T(G))]'$. Then $[A,G]' \subseteq N$. The result follows as above. □

When $G$ and $H$ are both torsion groups we have the following result:

Theorem 4.3.3. Suppose $G$ and $H$ are torsion groups such that $\mathbb{Z}G \simeq \mathbb{Z}H$. Then we have

i) $\gamma_i(G)/\gamma_{i+j}(G) \simeq \gamma_i(H)/\gamma_{i+j}(H)$ for $1 \leq j \leq i + 2$,

ii) $D_i(G)/D_{i+j}(G) \simeq D_i(H)/D_{i+j}(H)$ for $1 \leq j \leq i + 2$,

iii) $\delta_i(G)/\delta_{i+1}(G) \simeq \delta_i(H)/\delta_{i+1}(H)$ for all $i$,

iv) $\delta_i(G)/[G,\delta_i(G)]' \simeq \delta_i(H)/[H,\delta_i(H)]'$.

PROOF: Since (ii), (iii) and (iv) are immediate corollaries of theorem 4.3.2, it only remains to show (i). If $G$ and $H$ are torsion, then $\gamma_i(G) = T_i(G)$ and $\gamma_i(H) = T_i(H)$.

By applying theorem 4.3.2 (i), we have our result. □

Furukawa [9] proved (1) and (2) with $1 \leq j \leq i$. He also proved (3). The next result was proved by Ritter-Sehgal [15] with the

restriction that $G'$ is of exponent $p$.

Corollary 4.3.4. If $G$ and $H$ are torsion groups with $D_6(G) = 1$ and $\mathbb{Z}G \simeq \mathbb{Z}H$. Then $G' \simeq H'$.

We close with the famous theorem of Whitcomb's [20].

Corollary 4.3.5. Let $G$, $H$ be torsion groups. If $\mathbb{Z}G \simeq \mathbb{Z}H$ then

$$G/\delta_3(G) \simeq H/\delta_3(H).$$

## REFERENCES


[1] Bass, H.: Euler characteristics and characters of discrete groups, Invent. Math. 35(1976), 155-196.

[2] Bergman, G.M. and Dicks, W.: On universal derivations, J. Algebra 36(1975), 193-211.

[3] Bovdi, A.A.: Group rings of torsion free groups, Sibirsk. Math. Zh 1(1960), 555-558.

[4] Cliff, G.H., Sehgal, S.K., and Weiss, A.R.: Units of integral group rings of metabelian groups, J. Algebra 73(1981), 167-185.

[5] Cohn, J.A. and Livingstone, D.: On the structure of group algebras, Can. J. Math. 17(1965), 583-593.

[6] Dade, E.C.: Deux groupes finis distincts ayant la même algèbre de groupe sur toutcorps, Math. Z. 119(1971), 345-348.

[7] Fuchs, L.: Infinite abelian groups, Vol. Academic Press, New York, (1970).

[8] Furukawa, T.: Note on groups with isomorphic group algebras, Math. J. Okayama Univ. 24(1982), 1-6.

[9] Furukawa, T.: On isomorphism invariants of integral group rings, Math. J. Okayama Univ., 23(1981), 125-130.

[10] Hartley, B.: The residual nilpotence of wreath products, Proc. London Math. Soc. (3), 20(1970), 365-392.

[11] Losey, G.: On the structure of $Q_2(G)$ for finitely generated groups, Canad. J. Math. 25(1973), 353-359.

[12] Passi, I.B.S.: Group rings and their augmentation ideals, 715 Springer-Verlag, Berlin, Heidelberg, New York.

[13] Passmann, D.S.: Isomorphic groups and group rings, Pac. J. Math. 15(1965), 561-583.

[14] Passman, D.S. and Smith P.F.: Units in integral group rings, J. Algebra 69(1981), 213-239.

[15] Ritter, J. and Sehgal, S.K.: Isomorphism of group rings, Archiv. Der Mathematik, 40(1983), 32-39.

[16] Robinson, D.J.S.: Finiteness conditions and generalized soluble groups, Springer-Verlag, Berlin, Heidelberg, New York, 1(1972).

[17] Röhl, : Remarks on augmentation ideals (to appear).

[18] Sehgal, S.K.: Certain algebraic elements in group rings, Arch. Math. 26(1975), 139-143.

[19] Sehgal, S.K.: Topics in group rings, Dekker, New York.

[20] Whitcomb, A.: The group ring problem, Ph.D. Thesis, University of Chicago (1968).

[21] Ziegenbalg, J.: The lattices of the finite normal subgroups in group bases of isomorphic integral group rings are isomorphic, Arch. Math. 26(1975), 231-233.